J. Fluid Mech. (1980), *vol.* 99, *part* 3, *pp.* 481–493
Printed in Great Britain

Small-scale magnetic fields in turbulence: Saffman's approximation revisited

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(Received 26 July 1979)

The subject is the small-scale structure of a magnetic field in a turbulent conducting fluid, 'small scale' meaning lengths much smaller than the characteristic dissipative length of the turbulence. Philip Saffman developed an approximation to describe this structure and its evolution in time. Its usefulness invites a closer examination of the approximation itself and an attempt to place sharper limits on the numerical parameters that appear in the approximate correlation functions, topics to which the present paper is addressed.

A Lagrangian approach is taken, wherein one makes a Fourier decomposition of the magnetic field in a neighbourhood that follows a fluid element. If one construes the viscous–convective range narrowly, by ignoring magnetic dissipation entirely, then results for a magnetic field in two dimensions are consistent with Saffman's approximation, but in three dimensions no steady state could be found. Thus, in three dimensions, turbulent amplification seems to be more effective than Saffman's approximation implies. The cause seems to be a matter of geometry, not of correlation times or relative time scales.

Strictly-outward spectral transfer is a characteristic of Saffman's approximation, and this may be an accurate description only when dissipation suppresses the contributions from inwardly directed spectral transfer. In the spectral region where dominance passes from convection to dissipation, one can generate expressions for the parameters that arise in Saffman's approximation. Their numerical evaluation by computer simulation may enable one to sharpen the limits that Saffman had already set for those parameters.

1. Introduction

In astrophysics one encounters magnetic fields with a vast variety of length scales. This is true not only in absolute terms but also in relative terms: relative to a dissipative cut-off in a turbulence spectrum, the magnetic field may have a much larger scale or a smaller one. No recitation of the varieties is appropriate here, but the specific astrophysical scene that led to this paper deserves description, for it helps to set the mathematical context.

In the early stages of an expanding universe, turbulence can generate magnetic fields spontaneously (Harrison 1973). Such seed fields can be amplified throughout the radiation era, the period when the temperature still exceeds 4000 K and the conductivity of the cosmic plasma is exceedingly high. The kinematic viscosity increases by many orders of magnitude during this era, increasing the cutoff length of the turbulence

0022-1120/80/4489-1300 $02.00

and suggesting that the magnetic spectrum will be left behind at smaller length scales. Toward the close of the radiation era, the ratio of magnetic diffusivity to kinematic viscosity is less than 10^{-20}, and so the magnetic spectrum can extend far beyond the dissipative cut-off in the turbulence spectrum. The magnetic field is located predominantly in the viscous–convective range, viscous for the turbulence but convective for the field. In the expanding universe one needs to follow the evolution of such a field; results for a temporally stationary situation, even if reliable ones were available, would not suffice. An approximation developed by Philip Saffman (1963, 1964) can be applied to this problem (Baierlein 1978) and to other astrophysical contexts as well. Such usefulness invites a closer examination of the approximation itself and an attempt to place sharper limits on the numerical coefficients that appear in the approximate correlation functions. The present paper reports some contributions along these lines.

2. Mathematical framework

At this point one could display Saffman's approximation – indeed, the reader may wonder why that wasn't done already in the introduction – but it will be useful to develop a mathematical context for the display and then to derive Saffman's approximation by a variant of the routes that Saffman himself used.

The basic differential equation for a magnetic field in a turbulent conducting fluid is

$$\partial \mathbf{B}/\partial t = \operatorname{curl}(\mathbf{v} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B}, \tag{2.1}$$

where λ is the magnetic diffusivity, essentially the inverse of the electrical conductivity (Moffatt 1978). Both the magnetic field and the velocity field are specified to have zero divergence:

$$\operatorname{div} \mathbf{B} = 0, \quad \operatorname{div} \mathbf{v} = 0. \tag{2.2}$$

The objective is to analyse a small-scale magnetic field, a field whose dominant length scale is much smaller than the dissipative cut-off length of the turbulence. A spatial Fourier transform will bring the relative length scales into prominence. So let

$$B_j(\mathbf{x}, t) = (2\pi)^{-3} \int d^3k \, b_j(\mathbf{k}, t) \, e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{2.3}$$

and let $v_j(\mathbf{k}, t)$ denote the analogous Fourier coefficient of the velocity field. The Fourier transform of (2.1) can then be written

$$\partial b_j(\mathbf{k}, t)/\partial t = -i(2\pi)^{-3} \int d^3k' \, \{v_j(\mathbf{k}')\, k'_m \, b_m(\mathbf{k}-\mathbf{k}') - v_m(\mathbf{k}')\, k_m \, b_j(\mathbf{k}-\mathbf{k}')\} - \lambda k^2 b_j(\mathbf{k}), \tag{2.4}$$

where (2.2) has been used several times. The integral over $\mathbf{k}'$ cuts off sharply when k' exceeds k_d, the dissipative cut-off in the turbulence spectrum. Since we are concerned with the magnetic field at wavenumbers $k \gg k_d$, the argument $\mathbf{k}-\mathbf{k}'$ in the magnetic coefficients is always close to $\mathbf{k}$. Some expansion of those coefficients about $b_m(\mathbf{k})$ ought to be possible and fruitful.

At this point one must acknowledge the advective influence of the large turbulent eddies. Advection can produce rapid changes in the phase of a Fourier coefficient while leaving the amplitude substantially unchanged. Such phase changes depend sensitively on the full argument $\mathbf{k}-\mathbf{k}'$, and so no simple Taylor expansion of the coefficient is adequate. But one can cope with this problem. Precisely because we are concerned with the small-scale magnetic field, we can analyse the field in the neighbourhood of a single fluid element as that element is advected through space. The behaviour of $\mathbf{B}$ over distances that exceed the neighbourhood of the fluid element is not of interest. For the chosen element we can explicitly incorporate the phase change due to advection before attempting an expansion. And at some later stage we can average over many elements and their neighbourhoods.

So we shift our attention to a single fluid element and its neighbourhood. Let the position of that element be denoted by $\mathbf{y}(\mathbf{a}, t)$, where $\mathbf{a}$ is the position at time $t = 0$ and where

$$d\mathbf{y}(\mathbf{a}, t)/dt = \mathbf{v}(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{y}}. \tag{2.5}$$

Advection by the large eddies appears in a Fourier coefficient via a factor $\exp[+\mathbf{k}.\mathbf{y}]$. Thus a function like

$$b_m(\mathbf{k}-\mathbf{k}')\exp[-(\mathbf{k}-\mathbf{k}').\mathbf{y}]$$

is substantially independent of advection and can be expanded successfully in a Taylor's series in $\mathbf{k}'$. Such an expansion in (2.4) enables one to derive a tractable equation for the Fourier density tensor $D_{mj}(\mathbf{k}, t)$, defined by

$$D_{mj}(\mathbf{k}, t) \equiv b_m(\mathbf{k}, t)\, b_j^*(\mathbf{k}, t). \tag{2.6}$$

That equation is

$$\partial D_{mj}(\mathbf{k}, t)/\partial t = T_{mn} D_{nj} + T_{jn} D_{mn} - \partial[-k_a T_{an} D_{mj}]/\partial k_n - 2\lambda k^2 D_{mj}, \tag{2.7}$$

where

$$T_{mn}(t) \equiv \frac{\partial}{\partial x_n} v_m(\mathbf{x}, t)|_{\mathbf{x}=\mathbf{y}} \tag{2.8}$$

is the velocity-gradient tensor evaluated at the current location of the fluid element. Thus T_{mn} is a Lagrangian velocity-gradient tensor. The expansion in (2.4) was carried through terms linear in k'; discarding terms of order $(k')^2$ and higher amounts to discarding spatial variation in the local velocity gradient.

Although only an approximation, equation (2.7) is for us the basic working differential equation in wavenumber space. The change of focus to a single fluid element and its neighbourhood can be implemented in another, equivalent manner. One can introduce a new independent position variable $\boldsymbol{\xi}$,

$$\boldsymbol{\xi} \equiv \mathbf{x} - \mathbf{y}(\mathbf{a}, t),$$

so that spatial locations are reckoned relative to the instantaneous position of the fluid element (Hill & Bowhill 1978). Equation (2.1) can be converted to partial derivatives with respect to t and $\boldsymbol{\xi}$. The effective velocity field will become $\mathbf{v}(\mathbf{x}, t) - \mathbf{v}(\mathbf{y}, t)$, and thus advection is removed in the neighbourhood of the fluid element. A Fourier decomposition of the local magnetic field with respect to the variable $\boldsymbol{\xi}$ will lead to a differential equation like (2.4), but one in which a Taylor expansion is immediately permissible because advection has already been acknowledged.

The Fourier density tensor carries more information than one really cares about. In particular, one can average over the direction $\hat{\mathbf{k}}$ at fixed magnitude k and also form the trace. Applying those two operations to (2.7) yields

$$\frac{\partial}{\partial t}\langle D_{jj}\rangle_{\hat{\mathbf{k}}} = 2\langle T_{(mn)} D_{mn}\rangle_{\hat{\mathbf{k}}} - \frac{1}{k^{d-1}}\frac{\partial}{\partial k}[-k^d\langle \hat{\mathbf{k}}.\mathbf{T}.\hat{\mathbf{k}} D_{jj}\rangle_{\hat{\mathbf{k}}}] - 2\lambda k^2 \langle D_{jj}\rangle_{\hat{\mathbf{k}}}, \tag{2.9}$$

where

$$d = \text{spatial dimension} = 2 \text{ or } 3. \tag{2.10}$$

Here and henceforth the subscripts on angular brackets indicate the quantity over which the averaging was performed. The first term on the right-hand side of (2.9) describes amplification; the second, spectral transfer; and the third, dissipation. Only the symmetric portion $T_{(mn)}$ of the velocity gradient tensor survives the operation on (2.7), and so the Lagrangian strain field is the directly relevant quantity.

Several difficulties arise when one tries to solve either (2.7) or (2.9) as a stochastic differential equation, wherein the tensor T_{mn} is taken as a stochastic variable. The primary source of difficulty is the coincidence of two vital time scales. The differential equations by themselves imply that D_{mj} changes on a time scale set by $1/O(T_{mn})$. But the Lagrangian velocity gradient itself changes on a time scale of just this order. In different words, the auto-correlation time of the Lagrangian velocity gradient is about the same as the time scale for changes in D_{mj}, and so no approximation based on a short auto-correlation time can be invoked.

The persistence of the Lagrangian velocity gradient can, however, be turned to advantage, as Batchelor (1959) showed for a scalar field and Saffman (1963, 1964) did for vector fields. One can reason that $T_{mn}(t)$ and $D_{mj}(\mathbf{k},t)$ will be well correlated (at least after some transient period has elapsed). If one averages (2.9) over many fluid elements and over an initial distribution of the magnetic field, the result may have the form

$$\frac{\partial}{\partial t}D_S(k,t) = 2\frac{\sigma}{\tau}D_S - \frac{1}{k^{d-1}}\frac{\partial}{\partial k}\left[k^d\frac{1}{\tau}D_S\right] - 2\lambda k^2 D_S, \tag{2.11}$$

where

$$D_S(k,t) \equiv \langle D_{jj}\rangle_{\hat{\mathbf{k}},\,\mathbf{b},\,\mathrm{turbulence}}, \tag{2.12}$$

$$1/\tau \equiv \text{positive constant} = O[T_{(mn)}], \tag{2.13}$$

$$\sigma \equiv \text{positive constant} = O(1). \tag{2.14}$$

Equation (2.11) is Saffman's approximation, written in wavenumber space, though the present 'derivation' does not do justice to the care that Saffman exercised in his various routes to the approximation. The original routes show that $1/\tau$ should be equal (in magnitude) to some average of the largest negative strain eigenvalue, while σ/τ represents an analogous average of the largest positive strain eigenvalue.

At this point the extended introduction ends, and one can turn to the major questions. What range of validity does the approximation (2.11) possess? Can one sharpen the numerical limits that Saffman placed on σ?

Some solutions to equations (2.7), (2.9) and (2.11) can be found and, by comparing the solutions, one may take some progress toward answering the questions. So we turn to those solutions.

3. Some general solutions

A formal general solution to (2.7) can be written in terms of time-ordered exponentials. The solution itself is

$$D_{mj}(\mathbf{k},t) = U_{mp}(t)\,U_{jq}(t)\,D_{pq}(\mathbf{k}\,.\mathbf{U},0)\exp\left[-2\lambda\int_0^t dt'\,W_{ab}(t;t')\,k_a k_b\right], \tag{3.1}$$

while the definitions of the auxiliary tensors are these:

$$U_{mp}(t;t') \equiv \left[\exp\int_{t'}^{t} dt''\, T(t'')\right]_{mp,\,\text{time-ordered}}$$

$$\equiv \delta_{mp} + \int_{t'}^{t} dt_1\, T_{mp}(t_1) + \int_{t'}^{t} dt_1\, T_{ma}(t_1) \int_{t'}^{t_1} dt_2\, T_{ap}(t_2) + \dots, \tag{3.2}$$

$$U_{mp}(t) \equiv U_{mp}(t;0), \tag{3.3}$$

$$W_{ab}(t;t') \equiv U_{ap}(t;t')\, U_{bp}(t;t'). \tag{3.4}$$

Saffman (1963) provided an initial-value solution for his approximation in three dimensions; the generalization to d dimensions is simply

$$D_S(k,t) = e^{(2\sigma-d)\,t/\tau} D_S(ke^{-t/\tau},0) \exp\left[-2\lambda k^2 q^2(t)\right], \tag{3.5}$$

where

$$q^2(t) \equiv \tfrac{1}{2}\tau\left[1-e^{-2t/\tau}\right]. \tag{3.6}$$

The structure of Saffman's solution suggests that an initial spectrum whose form is a specific monomial in k should produce a steady state, either when λ can be ignored or after a transient period has elapsed. This is perhaps too naive a view, but it does suggest that one explore monomial initial spectra for the exact solution (3.1) and see whether a steady state in the viscous–convective range can be found. There will be angular integrals to evaluate; those associated with two spatial dimensions are by far the easier, and so let us start with that restricted geometry.

4. Two spatial dimensions

Several averages can be performed in the course of searching for a statistically steady state. A convenient starting point is this one: we focus on a specific fluid element and a specific wave vector $\mathbf{k}$ but average over the magnetic Fourier coefficients $\mathbf{b}(\mathbf{k},0)$ that could be associated with them. If we imagine that the magnetic field was introduced into the turbulence at $t=0$, then the average of $D_{mj}(\mathbf{k},0)$ over $\mathbf{b}(\mathbf{k},0)$ must have the isotropic form

$$\langle D_{mj}(\mathbf{k},0)\rangle_{\mathfrak{b}} = (\delta_{mj} - \hat{k}_m \hat{k}_j)\frac{1}{(d-1)}\, f(k^2) \tag{4.1}$$

with some scalar function $f(k^2)$, which is numerically equal to the average of the initial trace. An average over magnetic fields that had been introduced much earlier, but into the same turbulence, would exhibit a correlation with the fluid field and hence not necessarily possess the isotropic form (4.1); more about that later.

What monomial should one try for the scalar $f(k^2)$? Saffman's analysis enables one to estimate σ and then, from (3.5), to narrow the range of choices. Here is how that works. The property $\operatorname{div}\mathbf{v} = 0$ implies $\operatorname{Tr}\mathbf{T} = 0$, where '$\operatorname{Tr}\mathbf{T}$' is the *trace* of matrix $\mathbf{T}$. In two dimensions there are, of course, only two strain eigenvalues, whose sum must therefore be zero. If the averages that define σ/τ and $1/\tau$ have the same weighting functions, then $(\sigma/\tau)+(-1/\tau)$ must vanish in two dimensions. This suggests $\sigma_{2\,\text{dim}} = 1$. If that value is correct, then the first factor in (3.5) does not change with time, and so an initial trace that is independent of k would be the appropriate choice. Thus one is led to try

$$f(k^2) = \text{constant}. \tag{4.2}$$

If one pools (3.1), (4.1) and (4.2) and sets $\lambda = 0$, one can do the angular integrals required in an average over $\hat{\mathbf{k}}$. One finds

$$\langle D_{jj}(\mathbf{k},t)\rangle_{\mathrm{b},\hat{\mathbf{k}}} = \det \mathbf{W}^{\frac{1}{2}}\langle D_{jj}(\mathbf{k},0)\rangle_{\mathrm{b}}, \tag{4.3}$$

$$2\langle T_{(mn)}D_{mn}\rangle_{\mathrm{b},\hat{\mathbf{k}}} = \frac{2\,\mathrm{Tr}\,(\mathbf{TW}^{\frac{1}{2}})}{\mathrm{Tr}\,(\mathbf{W}^{\frac{1}{2}})}\langle D_{jj}(\mathbf{k},t)\rangle_{\mathrm{b},\hat{\mathbf{k}}}, \tag{4.4}$$

$$-\langle \hat{\mathbf{k}}\,.\,\mathbf{T}\,.\,\hat{\mathbf{k}}D_{jj}\rangle_{\mathrm{b},\hat{\mathbf{k}}} = \frac{-\mathrm{Tr}\,(\mathbf{TW}^{-\frac{1}{2}})}{\mathrm{Tr}\,(\mathbf{W}^{-\frac{1}{2}})}\langle D_{jj}(\mathbf{k},t)\rangle_{\mathrm{b},\hat{\mathbf{k}}}. \tag{4.5}$$

The tensor $\mathbf{W}$, defined by

$$W_{ab} \equiv W_{ab}(t;0) = U_{ap}(t)\,U_{bp}(t), \tag{4.6}$$

is symmetric and positive definite; so its square root, $\mathbf{W}^{\frac{1}{2}}$, is well-defined.

Only the properties of $\mathbf{W}$ just mentioned and $d = 2$ are needed to compute (4.3)–(4.5) from (3.1), (4.1) and (4.2). Now, however, one can note that $\mathrm{Tr}\,\mathbf{T} = 0$ implies $\det \mathbf{U}(t) = 1$, and so

$$\det \mathbf{W} = 1, \tag{4.7}$$

independent of dimension. This information, together with (4.3), tells us that we have indeed found a steady state, at least as far as $\langle D_{jj}\rangle_{\mathrm{b},\hat{\mathbf{k}}}$ is concerned.

To interpret (4.4) and (4.5), we need to know more about the tensor $\mathbf{W}$. At $t = 0$, that tensor is simply the unit tensor. After one turnover time for the small eddies, one can anticipate $\mathbf{W} = O(e^{\pm 1}) = O(3 \text{ or } \frac{1}{3})$, say, subject to (4.7). As time goes on, the spread in the probability distribution for $\mathbf{W}$ or its eigenvalues $\{w_j\}$ grows. An analysis by Cocke (1971) suggests that the spread associated with $\ln w_j$ may go as $t^{\frac{1}{2}}$. A cumulant expansion suggests

$$\langle W_{ab}(t)\rangle_{\mathrm{turbulence}} \sim \exp\,[\mathrm{const.} \times t],$$

subject to (4.7), as do analyses by Lumley (1972, 1978). Fortunately, specific forms are unimportant: what matters is that the typical eigenvalues quickly become large or small relative to unity, although their product is constrained by (4.7) to be precisely unity.

Since $\mathbf{W}$ is determined by an integral over $\mathbf{T}$, one expects $\mathbf{W}$ and $\mathbf{T}$ to be well-correlated. Then $\mathrm{Tr}\,(\mathbf{TW}^{\frac{1}{2}})$, which is dominated by the large eigenvalue of $\mathbf{W}$, is likely to be positive and of order 'positive strain eigenvalue' times $\mathrm{Tr}\,(\mathbf{W}^{\frac{1}{2}})$. But $\mathrm{Tr}\,(\mathbf{TW}^{-\frac{1}{2}})$ will be dominated by the small eigenvalue of $\mathbf{W}$, and so one can expect that trace to be of order 'negative strain eigenvalue' times $\mathrm{Tr}\,(\mathbf{W}^{-\frac{1}{2}})$. Saffman's factorization of the correlation functions on the left in (4.4) and (4.5) is supported, as is his qualitative description of the factors.

Next, one can form the ratio of (4.4) and (4.5) to extract a value for σ. Upon invoking $\mathrm{Tr}\,\mathbf{T} = 0$, one finds

$$\frac{2\langle T_{(mn)}D_{mn}\rangle_{\mathrm{b},\hat{\mathbf{k}}}}{-\langle \hat{\mathbf{k}}\,.\,\mathbf{T}\,.\,\hat{\mathbf{k}}D_{jj}\rangle_{\mathrm{b},\hat{\mathbf{k}}}} = +2, \tag{4.8}$$

independent of $\mathbf{W}$ and $\mathbf{T}$, and so a comparison of (2.9) and (2.11) implies

$$\sigma_{2\,\mathrm{dim}} = 1. \tag{4.9}$$

Auspicious.

Thus far we have worked with $f(k^2) = \text{constant}$. If, instead, one tries $f(k^2) \propto (k^2)^p$, with p some exponent near zero, say, $-0{\cdot}4 \leqslant p \leqslant +0{\cdot}4$, then one finds that $\langle D_{jj}\rangle_{\mathrm{b},\hat{\mathbf{k}}}$ increases with time if $p < 0$ and decreases if $p > 0$. The behaviour of Saffman's approximation is qualitatively the same.

5. Three spatial dimensions

The situation in three spatial dimensions is more complicated – by far. The property $\mathrm{Tr}\,\mathbf{T} = 0$ now says merely that the sum of three strain eigenvalues is zero and thus no longer gives a sharp estimate of what σ should be. An analysis by Saffman (1963, 1964) led to $\sigma \simeq 0{\cdot}8$ and almost surely $\sigma < 1$. One is inclined to set

$$f(k^2) \propto (k^2)^p \tag{5.1}$$

in (4.1) and look for a value of p that will yield a steady state for a suitable average of $D_{jj}(\mathbf{k}, t)$. From (3.5) one would infer $2\sigma - 3 - 2p = 0$, and therefore Saffman's estimates would suggest $p \simeq -0{\cdot}7$ and $p < -0{\cdot}5$.

So one combines (3.1), (4.1) and (5.1), together with $\lambda = 0$, and then one asks, can one choose p such that

$$I(\mathbf{W}; p) \equiv \langle D_{jj}(\mathbf{k}, t)\rangle_{\mathbf{b}, \hat{\mathbf{k}}}\big|_{\lambda=0} \tag{5.2}$$

remains steady as typical values of $\mathbf{W}$ increase, subject to $\det \mathbf{W} = 1$? Unfortunately, it seems that $I(\mathbf{W}; p)$ has an absolute minimum at $W_{ab} = \delta_{ab}$ for all reasonable values of p and that its growth as $\mathbf{W}$ becomes large is indeed a large growth, regardless of p. The bases for this conclusion are the following.

(1) An analysis of three limiting situations,

$$w_1 = w_2 \gg 1,$$

$$w_2 = w_3 \ll 1,$$

$$w_1 \gg w_2 = 1 \gg w_3,$$

as order-of-magnitude estimates for general p and as explicit integrations for tractable specific values of p.

(2) A variational calculation in the neighbourhood of $W_{ab} = \delta_{ab}$.

(3) Evaluation of $I(\mathbf{W}; p)$ by computer for $-1{\cdot}5 \leqslant p \leqslant 0{\cdot}5$ in steps of $\Delta p = 0{\cdot}1$ or $0{\cdot}2$ for a variety of sets $\{w_j\}$.

In short, in the sense that p would provide a numerical value for σ, there is no σ.

In two dimensions everything worked so well. Why this failure in three dimensions? Balancing amplification and spectral transfer in the viscous–convective range is a geometrically subtle matter, as we shall see explicitly later on. The addition of a third dimension – and hence a third strain eigendirection – increases the orientational freedom of magnetic field and wave vector relative to the strain field. This extra freedom upsets the precarious balance that can exist in two dimensions.

A scalar field, one should note, does not suffer such a qualitative change on the transition from two to three dimensions. Equations (2.7) and (2.9) can be adapted to a scalar field by deleting the amplification terms and by regarding D^T_{mj} as the scalar field. The analogue of (3.1) follows when the initial $\mathbf{U} \times \mathbf{U}$ factors are deleted and a trace is taken. An initial scalar spectrum of the monomial form, $\propto k^{-d}$, leads to angular averages that can be performed analytically in both two and three dimensions. One finds

$$\langle \text{scalar}\,(\mathbf{k}, t)\rangle_{\hat{\mathbf{k}}}\big|_{\lambda=0} = (\det \mathbf{W})^{-\frac{1}{2}}\langle \text{scalar}\,(\mathbf{k}, 0)\rangle_{\hat{\mathbf{k}}}, \tag{5.3}$$

and so $\det \mathbf{W} = 1$ implies constancy in time. The spectral form in three dimensions, $\propto k^{-3}$, was found by Batchelor (1959) and has reappeared in a variety of derivations.

Equation (5.3) provides one reason why that spectral form is so durable: it generates a steady state even without any averaging over the turbulence, that is, over the Lagrangian strain field and its history. In particular, that spectral form must be unaffected by intermittency.

No suitable exponent p or parameter σ could be defined. What is one to conclude from this? One must remember that the analysis was restricted to a narrowly construed viscous–convective range: the dissipation associated with λ was explicitly and entirely omitted. Within that rigid context, however, a conclusion would seem to be this: in three dimensions, amplification is more effective than Saffman's approximation implies.

The cause seems to be geometry, not correlation times. Support for this proposition comes from calculations done in the limit of short auto-correlation times. Such a limit conflicts with the persistence that Batchelor and Saffman believe characterizes the Lagrangian processes – as noted in § 2 – but the limit is instructive nonetheless. Details are left to the appendix. Suffice it to say here that the steady state in two dimensions is reproduced but, in three dimensions, no meaningful steady-state solution exists.

6. A kinematic paradox

The negative results in § 5 prompt one to ask, can one find a context where Saffman's approximation is manifestly valid. Examining spectral transfer will provide a clue.

If we look at the arguments of D_S in (3.5), we see that the solution to the approximation describes a spectral transfer that is strictly outward in wavenumber space. The formal solution (3.1), however, describes a more complicated spectral transfer. If we denote by $\mathbf{k}_{\text{old}}$ the wave vector at $t = 0$ that becomes $\mathbf{k}_{\text{new}}$ at time t, we can extract the relation

$$\mathbf{k}_{\text{old}} = \mathbf{k}_{\text{new}} . \mathbf{U}. \tag{6.1}$$

Multiplying from the right by $\mathbf{U}^{-1}$ and then squaring, we get

$$\begin{aligned} k^2_{\text{new}} &= \mathbf{k}_{\text{old}} . \mathbf{U}^{-1} . (\mathbf{U}^{-1})^T . \mathbf{k}_{\text{old}} \\ &= k^2_{\text{old}}\, \hat{\mathbf{k}}_{\text{old}} . \mathbf{W}^{-1} . \hat{\mathbf{k}}_{\text{old}}. \end{aligned} \tag{6.2}$$

So long as $\mathbf{W}^{-1}$ has at least one large eigenvalue, most orientations for $\hat{\mathbf{k}}_{\text{old}}$ will yield

$$k_{\text{new}} > k_{\text{old}} \tag{6.3}$$

as $\hat{\mathbf{k}}_{\text{old}}$ goes over 4π solid angle at fixed k^2_{old}. Thus most old wave vectors will be pushed outward in wavenumber space. This, of course, accords with the effects of persistent straining as envisaged by Batchelor and Saffman.

When one averages a quantity like $D_{jm}(\mathbf{k}, t)$ over $\hat{\mathbf{k}}$, however, one requires $\mathbf{k}_{\text{new}}$ to lie on the surface of a fixed sphere. One can ask where each $\mathbf{k}_{\text{new}}$ came from, in particular, what was k^2_{old} as a function of $\hat{\mathbf{k}}_{\text{new}}$ at fixed k^2_{new}. To answer that question, one need only square (6.1):

$$k^2_{\text{old}} = k^2_{\text{new}}\, \hat{\mathbf{k}}_{\text{new}} . \mathbf{W} . \hat{\mathbf{k}}_{\text{new}}. \tag{6.4}$$

Thus, provided $\mathbf{W}$ has at least one large eigenvalue, most orientations $\hat{\mathbf{k}}_{\text{new}}$ will imply

$$k_{\text{old}} > k_{\text{new}}. \tag{6.5}$$

And now we have a paradox.

The condition needed to generate the paradox is almost surely met. The requirement is simply that $\mathbf{W}$ have an eigenvalue that differs substantially from unity. Then $\det \mathbf{W} = 1$ will imply that there are at least two such eigenvalues, one small and one large, and they will do the job.

Let's look back a moment to the exact two-dimensional analysis in §4. Equation (4.5), when multiplied by k, gives the angular mean of the radial flux. We reasoned that $\mathrm{Tr}\,(\mathbf{TW}^{-\frac{1}{2}})$ would be negative and so the net flux would be positive, in accord with Saffman's picture. That reasoning is probably correct, but now we see that the net outward flux arises from a subtle balancing of outward and inward flow, wherein the inward contributions are – according to (6.5) – actually the more numerous, though not dominant.

For Saffman's approximation to be valid, it is necessary that angular mean values be overwhelmingly dominated by contributions which have evolved from smaller wavenumbers. That, in turn, requires strong variation of $\langle D_{mj}(\mathbf{k},t)\rangle_{\mathfrak{b}}$ with $\hat{\mathbf{k}}$ at fixed k. A narrowly construed viscous–convective range does not provide sufficiently strong variation, but, as one goes out in wavenumber space, dissipation begins to exert an influence that cannot be ignored. The dissipative exponential in (3.1) and, perhaps, an analogous exponential in the initial spectrum will cause the integrand in an angular average to peak around contributions that evolved from smaller wavenumber. If the peaking is sharp enough, it will produce the effectively one-way spectral transfer envisaged by Batchelor and Saffman. The importance of going to large k – and the need, perhaps – was recognized by Saffman (1963, p. 560).

When the integrand in an angular average is sharply peaked, one can generate some estimates for the parameters τ and σ, as follows. Equations (3.1) and (4.1) imply

$$\langle D_{mj}(\mathbf{k},t)\rangle_{\mathfrak{b}} = \left[W_{mj} - \frac{(\hat{\mathbf{k}}\,.\,\mathbf{W})_m(\hat{\mathbf{k}}\,.\,\mathbf{W})_j}{\hat{\mathbf{k}}\,.\,\mathbf{W}\,.\,\hat{\mathbf{k}}}\right] F(\mathbf{k},t) \tag{6.6}$$

with
$$F(\mathbf{k},t) \equiv \frac{1}{(d-1)} f(\mathbf{k}\,.\,\mathbf{W}\,.\,\mathbf{k}) \exp\left[-2\lambda \int_0^t dt'\, W_{ab}(t;t')\, k_a k_b\right]. \tag{6.7}$$

Suppose (2.9) is averaged over initial values of the magnetic field. The ensuing amplification term can be written as

$$2\langle T_{(mn)} D_{mn}\rangle_{\mathfrak{b},\hat{\mathbf{k}}} = 2\left\langle\left[\mathrm{Tr}\,(\mathbf{TW}) - \frac{\hat{\mathbf{k}}\,.\,\mathbf{W}\,.\,\mathbf{T}\,.\,\mathbf{W}\,.\,\hat{\mathbf{k}}}{\hat{\mathbf{k}}\,.\,\mathbf{W}\,.\,\hat{\mathbf{k}}}\right] F\right\rangle_{\hat{\mathbf{k}}}, \tag{6.8}$$

and this is to be compared with

$$\langle D_{jj}\rangle_{\mathfrak{b},\hat{\mathbf{k}}} = \left\langle\left[\mathrm{Tr}\,\mathbf{W} - \frac{\hat{\mathbf{k}}\,.\,\mathbf{W}^2\,.\,\hat{\mathbf{k}}}{\hat{\mathbf{k}}\,.\,\mathbf{W}\,.\,\hat{\mathbf{k}}}\right] F\right\rangle_{\hat{\mathbf{k}}}, \tag{6.9}$$

If the angular averages in these two expressions are dominated by those $\hat{\mathbf{k}}$ for which $\hat{\mathbf{k}}\,.\,\mathbf{W}\,.\,\hat{\mathbf{k}}$ is small, say, relative to $\mathrm{Tr}\,\mathbf{W}^T$, then the second terms should be negligible relative to the first, and one can extract the relation

$$2\langle T_{(mn)} D_{mn}\rangle_{\mathfrak{b},\hat{\mathbf{k}}} \simeq \frac{2\,\mathrm{Tr}\,(\mathbf{TW})}{\mathrm{Tr}\,\mathbf{W}} \langle D_{jj}\rangle_{\mathfrak{b},\hat{\mathbf{k}}}. \tag{6.10}$$

Next, one can examine the mean radial flux, which appears in the second term of (2.9). Suppose that D_{jj}, regarded as a function of $\hat{\mathbf{k}}$, peaks when $\hat{\mathbf{k}}\,.\,\mathbf{W}\,.\,\hat{\mathbf{k}}$ has its smallest value. It will be useful to re-express that condition in terms of the tensor $\mathbf{W}^{-\frac{1}{2}}$ and its principal axes. So let $\hat{\mathbf{k}}_{(a)}$ denote the normalized eigenvectors of $\mathbf{W}^{-\frac{1}{2}}$:

$$\hat{\mathbf{k}}_{(a)}.\mathbf{W}^{-\frac{1}{2}} = w_{(a)}^{-\frac{1}{2}}\hat{\mathbf{k}}_{(a)}, \tag{6.11}$$

with $a = 1$ labelling the largest eigenvalue of $\mathbf{W}^{-\frac{1}{2}}$. Then one has

$$\begin{aligned} -\langle\hat{\mathbf{k}}.\mathbf{T}.\hat{\mathbf{k}}D_{jj}\rangle_{\mathfrak{b},\hat{\mathbf{k}}} &\simeq -\hat{\mathbf{k}}_{(1)}.\mathbf{T}.\hat{\mathbf{k}}_{(1)}\langle D_{jj}\rangle_{\mathfrak{b},\hat{\mathbf{k}}} \\ &\simeq -\frac{\hat{\mathbf{k}}_{(1)}.\mathbf{W}^{-\frac{1}{2}}.\mathbf{T}.\mathbf{W}^{-\frac{1}{2}}.\hat{\mathbf{k}}_{(1)}}{[w_{(1)}^{-\frac{1}{2}}]^2}\langle D_{jj}\rangle_{\mathfrak{b},\hat{\mathbf{k}}} \end{aligned} \tag{6.12}$$

by (6.11). If $w_{(1)}^{-\frac{1}{2}}$ is much larger than the other two eigenvalues, then the denominator in (6.12) may be approximated by Tr $(\mathbf{W}^{-1})$. Similarly, the numerator may be extended to a sum over all three eigenvectors:

$$\begin{aligned} \hat{\mathbf{k}}_{(1)}.\mathbf{W}^{-\frac{1}{2}}.\mathbf{T}.\mathbf{W}^{-\frac{1}{2}}.\hat{\mathbf{k}}_{(1)} &\simeq \sum_a \hat{\mathbf{k}}_{(a)}.\mathbf{W}^{-\frac{1}{2}}.\mathbf{T}.\mathbf{W}^{-\frac{1}{2}}.\hat{\mathbf{k}}_{(a)} \\ &\simeq \mathrm{Tr}\,(\mathbf{TW}^{-1}); \end{aligned} \tag{6.13}$$

the last step follows because the set $\{\hat{\mathbf{k}}_{(a)}\}$ form an orthonormal triad. The upshot is

$$-\langle\hat{\mathbf{k}}.\mathbf{T}.\hat{\mathbf{k}}D_{jj}\rangle_{\mathfrak{b},\hat{\mathbf{k}}} \simeq -\frac{\mathrm{Tr}\,(\mathbf{TW}^{-1})}{\mathrm{Tr}\,(\mathbf{W}^{-1})}\langle D_{jj}\rangle_{\mathfrak{b},\hat{\mathbf{k}}}, \tag{6.14}$$

though one must concede immediately that one could carry through the same reasoning with almost any negative power of $\mathbf{W}$, and so the coefficient on the right-hand side may not be the optimum estimate.

In principle, one can determine the parameters τ and σ by comparing (2.11) with a suitable average of (2.9). This means one should really average (6.10) and (6.14) over many fluid elements before trying to extract τ and σ. That introduces correlations between the factors to which one cannot really do justice analytically. One might as well assess the parameters directly from a comparison among (2.9), (2.11), (6.10) and (6.14):

$$\sigma/\tau \simeq \frac{\mathrm{Tr}\,(\mathbf{TW})}{\mathrm{Tr}\,\mathbf{W}}, \tag{6.15}$$

$$1/\tau \simeq -\frac{\mathrm{Tr}\,(\mathbf{TW}^{-1})}{\mathrm{Tr}\,(\mathbf{W}^{-1})}. \tag{6.16}$$

Some average of the right-hand sides over turbulence is in order, but the estimates are meaningful only if averages do not differ much from typical values, in the root mean square sense, say. Moreover, sufficient time must have elapsed for typical $\mathbf{W}$'s to have reached values substantially different from the unit tensor.

The estimates (6.15) and (6.16) are rotationally invariant expressions, with natural weighting functions, for two strain values. Although their derivation leaves something to be desired, one may be able to calculate them numerically by computer simulation of turbulence and thereby gain a better estimate of the parameters, in particular, of the ratio σ.

7. Closing remarks

Some remarks to close out the discussion are in order.

Much of the preceding analysis used (4.1) as an integral element. That initial condition might be suspect on the grounds that correlations between **B** and the turbulence at $t = 0$ are inadequately described: a statistically steady state would possess correlations. But, provided one waits an eddy turnover time or two, that potential inadequacy should make no difference, as indeed seems to be true for the magnetic field in two dimensions (and for scalar fields in both two and three dimensions). For the magnetic field in three dimensions, one could ask – at the very least – why doesn't a natural form like (4.1) lead, after a transient period, to a statistically steady state? That it seems not to is itself significant.

The implications of §5, as expressed near the end of that section, bear reiterating. If one construes the viscous–convective range narrowly, by ignoring magnetic dissipation entirely, and if one works with a magnetic field in three dimensions, then turbulent amplification seems to be more effective than Saffman's approximation implies. The cause seems to be a matter of geometry, not of correlation times or relative time scales.

One has to bear in mind that the steps from (2.1) through (2.4) to (2.7) entailed two qualitatively major approximations: (1) a change of focus from the entire fluid to the neighbourhood of an individual fluid element and (2) an expansion (in effect) of the Lagrangian velocity field which stopped at the first derivative of that field. For studying the small-scale structure of the magnetic field, these approximations should be valid. Yet Saffman's comment, '...we must again conclude that the application of Batchelor's ideas to [vector fields] is not entirely equivalent to considering the behaviour of random distributions in straining motions of infinite extent' (1963, p. 556), gives one pause. It is conceivable that Saffman's approximation is valid in the viscous–convective range and yet one cannot show that with the route taken in this paper.

My thanks for hospitality go to Professor Ronald Ruby and the Physics Board at the University of California, Santa Cruz, where this work was completed during a sabbatical leave.

Appendix. The limit of a short auto-correlation time

Approximations for stochastic differential equations that are based on a short auto-correlation time have been reviewed very nicely by Van Kampen (1976). Kraichnan (1968) applied the idea to the small-scale structure of a scalar field in turbulence.

Our starting point is (2.7), expressed in operator form and with $\lambda = 0$:

$$\partial D_{mj}/\partial t = Q_{mjnp}\, D_{np}, \tag{A 1}$$

where

$$Q_{mjnp} \equiv a'\{\delta_{mn}\, T_{jp} + \delta_{jp}\, T_{mn}\} + T_{ab}\, k_a\, \delta_{mn}\, \delta_{jp}(\partial/\partial k_b). \tag{A 2}$$

The parameter a' is inserted to keep track of the amplification terms that operate for a vector field. A numerical value of $a' = 1$ gives that situation, while $a' = 0$, together with a trace operation, will generate the equation appropriate for a scalar field.

Now we form an average of (A 1) over a set of fluid elements and over initial values of the magnetic field:

$$\partial\langle D_{mj}\rangle/\partial t = \langle Q_{mjnp}\, D_{np}\rangle. \tag{A 3}$$

The operator $\mathbf{Q}$ is linear in the tensor $\mathbf{T}$, which changes on the time scale τ_c, its auto-correlation time. As before, the differential equation (A 1) says that D_{mj} changes on the time scale $1/O(T_{mn})$. One now *specifies* that $\tau_c \ll 1/O(T_{mn})$.

When the strong inequality between time scales holds, only 'a small portion' of $D_{np}(\mathbf{k}, t)$ can be well-correlated with $Q_{mjnp}(t)$, essentially the increment in D_{np} over the interval $t-\tau_c$ to t, which was produced by $\mathbf{T}$'s with which $\mathbf{T}(t)$ is well-correlated. This suggests writing D_{np} in (A 3) as the integral of its time derivative and using (A 1) again:

$$\partial\langle D_{mj}\rangle/\partial t = \langle Q_{mjnp}(t)\, D_{np}(\mathbf{k}, 0)\rangle + \int_0^t dt'\langle Q_{mjnp}(t)\, Q_{npab}(t')\, D_{ab}(\mathbf{k}, t')\rangle. \tag{A 4}$$

Once $t \gg \tau_c$, the first term can be set to zero because $\mathbf{Q}(t)$ will be uncorrelated with $D_{np}(\mathbf{k}, 0)$ and $\langle \mathbf{Q}(t)\rangle = 0$ because it is linear in the tensor $\mathbf{T}$. The integrand in the second term will be non-zero only over an interval $(t - \text{few}\,\tau_c) \lesssim t' \leqslant t$, and during this period D_{ab} changes little. Moreover, one can get a non-zero result even if one ignores the correlation between (a small portion of) D_{ab} and the operator product. (Correlation corrections can be generated in a systematic fashion, as Van Kampen 1976 describes so well.) Upon factoring the second term in (A 4), one arrives at

$$\partial\langle D_{mj}\rangle/\partial t = \int_0^t dt'\,\langle Q_{mjnp}(t)\, Q_{npab}(t')\rangle\,\langle D_{ab}(\mathbf{k}, t)\rangle. \tag{A 5}$$

The averages over turbulence and initial field imply that $\langle D_{mj}\rangle$ must have the isotropic form

$$\langle D_{mj}(\mathbf{k}, t)\rangle = [(\delta_{mj} - \hat{k}_m \hat{k}_j)/(d-1)]\, D(k^2, t). \tag{A 6}$$

The rest is a matter of algebra: one finds

$$\frac{\partial}{\partial t} D = C_1\left\{C_2 D + \frac{1}{k^{C_3-2}}\frac{\partial}{\partial k}\left(k^{C_3}\frac{\partial D}{\partial k}\right)\right\} \tag{A 7}$$

with

$$C_1 \equiv \int_0^t dt'\,\langle T_{(ab)}(t)\, T_{ab}(t')\rangle \times \begin{Bmatrix} \frac{1}{4} \\ \frac{2}{15} \end{Bmatrix}, \tag{A 8}$$

$$C_2 \equiv \begin{Bmatrix} 8a'(a'-1) \\ a'(10a'-6) \end{Bmatrix}, \tag{A 9}$$

$$C_3 \equiv \begin{Bmatrix} 3-4a' \\ 4-2a' \end{Bmatrix}, \tag{A 10}$$

where the upper entry refers to spatial dimension $d = 2$ and the lower to $d = 3$.

The vector field in two dimensions has the steady-state solutions

$$\langle D_{jj}\rangle_{2\,\mathrm{dim}} = \text{const., const.}' \times k^2. \tag{A 11}$$

Both of these spectral forms occur as steady solutions in (3.1) when that equation is averaged as in §4.

The situation in three dimensions is quite different. Trying for a steady state with a monomial leads to

$$\begin{aligned}\langle D_{jj}\rangle_{3\,\mathrm{dim}} &\propto k^s \quad \text{with} \quad s = -\tfrac{1}{2} \pm i(\tfrac{15}{4})^{\frac{1}{2}} \\ &= |\text{const.}| \times \left(\frac{k}{k_f}\right)^{-\frac{1}{2}} \cos\left\{\left(\frac{15}{4}\right)^{\frac{1}{2}} \ln\frac{k}{k_f} + \text{phase}\right\},\end{aligned} \tag{A 12}$$

where k_f is some fiducial wavenumber. Since (A 12) contains two linearly independent solutions, it is a general solution. The viscous–convective range can certainly be large enough for k/k_f to vary greatly, and so the right-hand side will be driven negative, no matter what value is chosen for the phase. This conflicts with the positive definiteness of the left-hand side, and so no meaningful steady-state solution exists.

An aside about the two-dimensional geometry is in order. The coefficient C_2 vanishes not only when $a' = 0$ but also when $a' = 1$, suggesting a closer link with the scalar field in two dimensions than exists in three dimensions. This is indeed so, as the following analysis shows.

If one thinks of the two-dimensional system as embedded in a three-dimensional space, in particular, as occupying the plane $z = 0$, then one can write

$$B_j = \epsilon_{jm3}\,\partial\mathscr{A}/\partial x_m, \quad v_j = \epsilon_{jm3}\,\partial\psi/\partial x_m, \tag{A 13}$$

where $\mathscr{A}\hat{\mathbf{z}}$ is a vector potential and ψ a stream function. Next, suppose one specifies that the 'scalar' $\mathscr{A}$ is to satisfy a convective, dissipative equation:

$$\partial\mathscr{A}/\partial t = -\mathbf{v}\,.\,\mathrm{grad}\,\mathscr{A} + \lambda\nabla^2\mathscr{A}. \tag{A 14}$$

Then operation on this equation with $\epsilon_{jm3}(\partial/\partial x_m)$, together with (A 13), will generate precisely the differential equation that emerges when the vector operations in (2.1) are expanded as if in three dimensions and the result is projected onto the (x, y) plane. The route can, of course, be reversed to deduce from (2.1) and (A 13) that (A 14) will hold.

Although a transverse vector field $\mathbf{B}$ must be generated from the 'scalar' $\mathscr{A}$, there is a certain strong sense in which the two-dimensional magnetic field is not essentially different from a scalar field, especially when one examines mean values in wavenumber space. This kinship goes a long way toward explaining why the two-dimensional magnetic field has steady-state behaviour as good as that for a *bona fide* scalar field.

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